

BRANCHING FORMULA FOR MACDONALD-KOORNWINDER POLYNOMIALS

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ABSTRACT. We present an explicit branching formula for the six-parameter Macdonald-Koornwinder polynomials with hyperoctahedral symmetry.

1. INTRODUCTION

Branching formulas constitute a powerful tool in algebraic combinatorics providing a recursive scheme to build symmetric polynomials via induction in the number of variables [M1, LW]. The combinatorial aspects of the hyperoctahedral-symmetric Macdonald-Koornwinder polynomials [M2, K] were explored in seminal works of Okounkov and Rains [O, R]. In particular, the structure of a branching formula for the Macdonald-Koornwinder polynomials has been outlined at the end of [R, Sec. 5]. The aim of the present note is to make the branching polynomials under consideration explicit. Following the ideas underlying the proof of the branching formula for the Macdonald polynomials [M1, Ch. VI.7], our main tools to achieve this goal consist of: Mimachi's Cauchy formula for the Macdonald-Koornwinder polynomials [M], (a special 'column-row' case of) the Cauchy formula for Okounkov's hyperoctahedral interpolation polynomials [O], and explicitly known Pieri coefficients for the Macdonald-Koornwinder polynomials [D2, D4].

The material is structured as follows. After recalling some necessary preliminaries regarding the Macdonald-Koornwinder polynomials and their Pieri formulas in Section 2, our branching formula is first stated in Section 3 and then proven in Section 4.

2. PRELIMINARIES

2.1. Macdonald-Koornwinder polynomials [K]. For a partition

$$\lambda \in \Lambda_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}, \quad (2.1)$$

the monic Macdonald-Koornwinder polynomial $P_\lambda(z_1, \dots, z_n; q, t, \mathbf{t})$ is a Laurent polynomial in the complex variables z_1, \dots, z_n that depends rationally on the parameters q, t and $\mathbf{t} := (t_0, t_1, t_2, t_3)$. It is determined by a leading monomial of the form $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_n^{\lambda_n}$, while being symmetric with respect to the action of the hyperoctahedral group $W = S_n \ltimes \mathbb{Z}_2^n$ by permutations and inversions of the variables. For

Date: August 2014.

2000 Mathematics Subject Classification. 33D52, 05E05.

Key words and phrases. Macdonald-Koornwinder polynomials, hyperoctahedral symmetric functions, branching rules.

This work was supported in part by the *Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT)* Grants # 1130226 and # 1141114.

real parameter values in the domain $0 < q, |t|, |t_l| < 1$ ($l = 0, 1, 2, 3$), the Macdonald-Koornwinder polynomials form an orthogonal system on the n -dimensional torus $|z_j| = 1, j = 1, \dots, n$. The orthogonality measure is given by Gustafson's q -Selberg type density [G]

$$\Delta = \prod_{1 \leq j \leq n} \frac{(z_j^2, z_j^{-2}; q)_\infty}{\prod_{0 \leq l \leq 3} (t_l z_j, t_l z_j^{-1}; q)_\infty} \prod_{1 \leq j < k \leq n} \frac{(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)_\infty}{(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1}; q)_\infty}$$

with respect to the Haar measure on this torus. Here and below we employ standard conventions for the q -Pochhammer symbols: $(a; q)_k := (1-a)(1-aq) \cdots (1-aq^{k-1})$ (with $(a; q)_0 := 1$) and $(a_1, \dots, a_l; q)_k := (a_1; q)_k \cdots (a_l; q)_k$.

2.2. Pieri coefficients [D2, D4, S]. The W -invariant Laurent polynomials

$$E_r(z_1, \dots, z_n; t, t_0) = \sum_{1 \leq j_1 < \dots < j_r \leq n} \langle z_{j_1}; t^{j_1-1} t_0 \rangle \cdots \langle z_{j_r}; t^{j_r-1} t_0 \rangle \quad (2.2)$$

($r = 1, \dots, n$), with $\langle z; x \rangle := z + z^{-1} - x - x^{-1}$, are special instances of Okounkov's hyperoctahedral interpolation polynomials [O]—with shifted variables as considered by Rains [R]—that correspond to the partitions with only a single column [KNS]. They describe the eigenvalues of commuting difference operators diagonalized by the Macdonald-Koornwinder polynomials [D1] and are also instrumental in Ito's Aomoto-style proof [I] of the hyperoctahedral ${}_6\Psi_6$ sum evaluated in Ref. [D3].

Let $C_{\lambda, r}^{\mu, n}(q, t, \mathbf{t})$ denote the coefficients in the Macdonald-Koornwinder Pieri-expansions associated with these one-column interpolation polynomials:

$$E_r(z_1, \dots, z_n; t, t_0) P_\lambda(z_1, \dots, z_n; q, t, \mathbf{t}) = \sum_{\substack{\mu \in \Lambda_n \\ \mu \sim_r \lambda}} C_{\lambda, r}^{\mu, n}(q, t, \mathbf{t}) P_\mu(z_1, \dots, z_n; q, t, \mathbf{t}), \quad (2.3)$$

$r = 1, \dots, n$. To filter only the nonvanishing coefficients, we have employed the following proximity relation within Λ_n restricting the sum on the RHS: $\mu \sim_r \lambda$ iff there exists a partition $\nu \in \Lambda_n$ with $\nu \subset \lambda$ and $\nu \subset \mu$ such that the skew diagrams λ/ν and μ/ν are vertical strips with $|\lambda/\nu| + |\mu/\nu| \leq r$. Here $|\cdot|$ refers to the number of boxes of the diagram, $\nu \subset \lambda$ means that $\nu \in \Lambda_n$ is contained in λ : $\nu_j \leq \lambda_j$ ($j = 1, \dots, n$), and (recall) the skew diagram λ/ν is a vertical strip iff $\nu_j \leq \lambda_j \leq \nu_j + 1$ ($j = 1, \dots, n$).

Upon writing $J = \{1 \leq j \leq n \mid \lambda_j \neq \mu_j\}$, $J^c = \{1, \dots, n\} \setminus J$, and $\epsilon_j = \mu_j - \lambda_j$ for $j \in J$ (so, if $\mu \sim_r \lambda$ the cardinality $|J|$ of J is at most r and $\epsilon_j \in \{1, -1\}$), one can express the Pieri coefficients in question explicitly as follows:

$$C_{\lambda, r}^{\mu, n}(q, t, \mathbf{t}) = \frac{P_\lambda(\hat{\tau}_1, \dots, \hat{\tau}_n; q, t, \mathbf{t})}{P_\mu(\hat{\tau}_1, \dots, \hat{\tau}_n; q, t, \mathbf{t})} V_{\epsilon J}^n(\lambda; q, t, \mathbf{t}) U_{J^c, r-|J|}^n(\lambda; q, t, \mathbf{t}), \quad (2.4)$$

where

$$P_\lambda(\hat{\tau}_1, \dots, \hat{\tau}_n; q, t, \mathbf{t}) = \prod_{1 \leq j \leq n} \frac{\prod_{0 \leq l \leq 3} (t_l \hat{\tau}_j; q)_{\lambda_j}}{\tau_j^{\lambda_j} (\hat{\tau}_j^2; q)_{2\lambda_j}} \prod_{1 \leq j < k \leq n} \frac{(t \hat{\tau}_j \hat{\tau}_k; q)_{\lambda_j + \lambda_k} (t \hat{\tau}_j \hat{\tau}_k^{-1}; q)_{\lambda_j - \lambda_k}}{(\hat{\tau}_j \hat{\tau}_k; q)_{\lambda_j + \lambda_k} (\hat{\tau}_j \hat{\tau}_k^{-1}; q)_{\lambda_j - \lambda_k}},$$

$$\begin{aligned}
V_{\epsilon J}^n(\lambda; q, t, \mathbf{t}) &= \prod_{j \in J} \frac{\prod_{0 \leq l \leq 3} (1 - \hat{t}_l \hat{\tau}_j^{\epsilon_j} q^{\epsilon_j \lambda_j})}{t_0 (1 - \hat{\tau}_j^{2\epsilon_j} q^{2\epsilon_j \lambda_j}) (1 - \hat{\tau}_j^{2\epsilon_j} q^{2\epsilon_j \lambda_j + 1})} \\
&\times \prod_{\substack{j, j' \in J \\ j < j'}} \frac{(1 - t \hat{\tau}_j^{\epsilon_j} \hat{\tau}_{j'}^{\epsilon_{j'}} q^{\epsilon_j \lambda_j + \epsilon_{j'} \lambda_{j'}}) (1 - t \hat{\tau}_j^{\epsilon_j} \hat{\tau}_{j'}^{\epsilon_{j'}} q^{\epsilon_j \lambda_j + \epsilon_{j'} \lambda_{j'} + 1})}{t (1 - \hat{\tau}_j^{\epsilon_j} \hat{\tau}_{j'}^{\epsilon_{j'}} q^{\epsilon_j \lambda_j + \epsilon_{j'} \lambda_{j'}}) (1 - \hat{\tau}_j^{\epsilon_j} \hat{\tau}_{j'}^{\epsilon_{j'}} q^{\epsilon_j \lambda_j + \epsilon_{j'} \lambda_{j'} + 1})} \\
&\times \prod_{j \in J, k \in J^c} \frac{(1 - t \hat{\tau}_j^{\epsilon_j} \hat{\tau}_k q^{\epsilon_j \lambda_j + \lambda_k}) (1 - t \hat{\tau}_j^{\epsilon_j} \hat{\tau}_k^{-1} q^{\epsilon_j \lambda_j - \lambda_k})}{t (1 - \hat{\tau}_j^{\epsilon_j} \hat{\tau}_k q^{\epsilon_j \lambda_j + \lambda_k}) (1 - \hat{\tau}_j^{\epsilon_j} \hat{\tau}_k^{-1} q^{\epsilon_j \lambda_j - \lambda_k})},
\end{aligned}$$

and

$$\begin{aligned}
U_{K,p}^n(\lambda; q, t, \mathbf{t}) &= (-1)^p \sum_{\substack{I \subset K, |I|=p \\ \epsilon_i \in \{1, -1\}, i \in I}} \left(\prod_{i \in I} \frac{\prod_{0 \leq l \leq 3} (1 - \hat{t}_l \hat{\tau}_i^{\epsilon_i} q^{\epsilon_i \lambda_i})}{t_0 (1 - \hat{\tau}_i^{2\epsilon_i} q^{2\epsilon_i \lambda_i}) (1 - \hat{\tau}_i^{2\epsilon_i} q^{2\epsilon_i \lambda_i + 1})} \right. \\
&\times \prod_{\substack{i, i' \in I \\ i < i'}} \frac{(1 - t \hat{\tau}_i^{\epsilon_i} \hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i \lambda_i + \epsilon_{i'} \lambda_{i'}}) (1 - t^{-1} \hat{\tau}_i^{\epsilon_i} \hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i \lambda_i + \epsilon_{i'} \lambda_{i'} + 1})}{(1 - \hat{\tau}_i^{\epsilon_i} \hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i \lambda_i + \epsilon_{i'} \lambda_{i'}}) (1 - \hat{\tau}_i^{\epsilon_i} \hat{\tau}_{i'}^{\epsilon_{i'}} q^{\epsilon_i \lambda_i + \epsilon_{i'} \lambda_{i'} + 1})} \\
&\times \left. \prod_{i \in I, k \in K \setminus I} \frac{(1 - t \hat{\tau}_i^{\epsilon_i} \hat{\tau}_k q^{\epsilon_i \lambda_i + \lambda_k}) (1 - t \hat{\tau}_i^{\epsilon_i} \hat{\tau}_k^{-1} q^{\epsilon_i \lambda_i - \lambda_k})}{t (1 - \hat{\tau}_i^{\epsilon_i} \hat{\tau}_k q^{\epsilon_i \lambda_i + \lambda_k}) (1 - \hat{\tau}_i^{\epsilon_i} \hat{\tau}_k^{-1} q^{\epsilon_i \lambda_i - \lambda_k})} \right)
\end{aligned}$$

for $p = 0, \dots, |K|$ (with the convention that $V_{\epsilon J}^n(\lambda; q, t, \mathbf{t}) = 1$ if J is empty and $U_{K,p}^n(\lambda; q, t, \mathbf{t}) = 1$ if $p = 0$). Here

$$\tau_j = t^{n-j} t_0, \quad \hat{\tau}_j = t^{n-j} \hat{t}_0 \quad (j = 1, \dots, n),$$

and

$$\hat{t}_0^2 = q^{-1} t_0 t_1 t_2 t_3, \quad \hat{t}_0 \hat{t}_l = t_0 t_l \quad (l = 1, 2, 3).$$

Remark 1. Below we will employ a trivially extended notion of $E_r(z_1, \dots, z_n; t, t_0)$ and $C_{\lambda, r}^{\mu, n}(q, t, \mathbf{t})$ that allows r and n to become equal to zero. By convention $E_0(z_1, \dots, z_n; t, t_0) := 1$, whence for the corresponding coefficients $C_{\lambda, 0}^{\mu, n}(q, t, \mathbf{t}) = 1$ if $\mu = \lambda$ and vanishes otherwise.

3. BRANCHING FORMULA

Let us recall that for $\mu \subset \lambda \in \Lambda_n$, the skew diagram λ/μ is a horizontal strip provided the parts of λ and μ interlace as follows:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_n \geq \mu_n.$$

We will need the following relation in Λ_n expressing that the partition λ can be obtained from $\mu \subset \lambda$ by adding at most two horizontal strips: $\mu \preceq \lambda$ iff there exists a $\nu \in \Lambda_n$ with $\mu \subset \nu \subset \lambda$ such that the skew diagrams λ/ν and ν/μ are horizontal strips. From now on we will think of Λ_n as being embedded in Λ_{n+1} in the natural way (i.e. ‘by adding a part of size zero’). The main result of this note is given by the following branching formula for the Macdonald-Koornwinder polynomials, the proof of which is delayed until Section 4 below.

We denote by $m^n \in \Lambda_n$ the rectangular partition such that $(m^n)_j = m$ ($j = 1, \dots, n$) and—more generally—by $m^n - \mu$ with $\mu \subset m^n$ the partition such that $(m^n - \mu)_j = m - \mu_{n+1-j}$ ($j = 1, \dots, n$). Finally, we write $\lambda' \in \Lambda_m$ ($m \geq \lambda_1$) for

the conjugate partition of $\lambda \in \Lambda_n$, i.e. with λ'_i counting the number of parts of λ that are greater or equal than i ($i = 1, \dots, m$).

Theorem 1 (Branching Formula). *For $\lambda \in \Lambda_{n+1}$, the Macdonald-Koornwinder polynomial in $(n+1)$ variables expands in terms of the n -variable polynomials as*

$$P_\lambda(z_1, \dots, z_n, x; q, t, \mathbf{t}) = \sum_{\substack{\mu \in \Lambda_n \\ \mu \preceq \lambda}} P_\mu(z_1, \dots, z_n; q, t, \mathbf{t}) P_{\lambda/\mu}(x; q, t, \mathbf{t}), \quad (3.1a)$$

with one-variable branching polynomials of degree $d = |\{1 \leq j \leq m \mid \lambda'_j = \mu'_j + 1\}|$ whose expansion

$$P_{\lambda/\mu}(x; q, t, \mathbf{t}) = \sum_{0 \leq k \leq d} B_{\lambda/\mu}^k(q, t, \mathbf{t}) \langle x; t_0 \rangle_{q,k} \quad (3.1b)$$

in the basis of the interpolation polynomials in one variable

$$\langle x; t_0 \rangle_{q,k} := \langle x; t_0 \rangle \langle x; qt_0 \rangle \cdots \langle x; q^{k-1} t_0 \rangle \quad (\text{with } \langle x; t_0 \rangle_{q,0} := 1) \quad (3.1c)$$

has coefficients that are given explicitly by the Macdonald-Koornwinder Pieri coefficients in $m = \lambda_1$ variables:

$$B_{\lambda/\mu}^k(q, t, \mathbf{t}) = (-1)^{k+|\lambda|-|\mu|} C_{n^m-\mu', m-k}^{(n+1)^m-\lambda', m}(t, q, \mathbf{t}) \quad (k = 0, \dots, d). \quad (3.1d)$$

It is well-known [K] that in the case of only a single variable the Macdonald-Koornwinder polynomials reduce to the five-parameter monic Askey-Wilson polynomials $P_m(z; q, \mathbf{t})$, $m = 0, 1, 2, \dots$ [AW]. On the other hand, if we formally set $n = 0$ and $\mu = 0$ then (the proof of) Theorem 1 remains valid. This gives rise to the following expansion of the Askey-Wilson polynomials in terms of the one-variable interpolation polynomials $\langle x; t_0 \rangle_{q,k}$.

Corollary 2 (Askey-Wilson Polynomials). *The monic Askey-Wilson polynomial of degree m is given by*

$$P_m(z; q, \mathbf{t}) = \sum_{0 \leq k \leq m} B_{m/0}^k(q, \mathbf{t}) \langle z; t_0 \rangle_{q,k} \quad (3.2a)$$

with

$$B_{m/0}^k(q, \mathbf{t}) = (-1)^{m+k} C_{0^m, m-k}^{0^m, m}(t, q, \mathbf{t}). \quad (3.2b)$$

This formula for the Askey-Wilson polynomials amounts to the $n = 1$ case of Okounkov's binomial formula for the Macdonald-Koornwinder polynomials [O, Thm. 7.1] with the binomial coefficients written explicitly in terms of the m -variable Macdonald-Koornwinder Pieri coefficients. Notice that since the Askey-Wilson polynomials on the LHS are independent of t , it follows that the t -dependence of the corresponding branching coefficients drops out as well. In this special situation, alternative expressions for the relevant binomial coefficients are available in a much more compact form [R, Prp. 4.1] and the binomial formula is in fact seen to reduce to the usual ${}_4\phi_3$ representation of the Askey-Wilson polynomial [KNS, p. 25].

By iterating the branching formula in Theorem 1, one finds the general Macdonald-Koornwinder branching polynomial as a sum of factorized contributions over ascending chains of partitions.

Corollary 3 (Branching Polynomials). *For $\lambda \in \Lambda_{n+l}$, one has that*

$$P_\lambda(z_1, \dots, z_n, x_1, \dots, x_l; q, t, \mathbf{t}) = \sum_{\substack{\mu^{(i)} \in \Lambda_{n+i}, i=0, \dots, l \\ \mu = \mu^{(0)} \preceq \mu^{(1)} \preceq \dots \preceq \mu^{(l)} = \lambda}} P_\mu(z_1, \dots, z_n; q, t, \mathbf{t}) \prod_{1 \leq i \leq l} P_{\mu^{(i)}/\mu^{(i-1)}}(x_i; q, t, \mathbf{t}). \quad (3.3)$$

Setting $n = 1$ in the latter formula, leads us to an explicit formula for the Macdonald-Koornwinder polynomials generalizing the formula for the Askey-Wilson polynomials in Corollary 2.

Corollary 4 (Macdonald-Koornwinder Polynomials). *For $\lambda \in \Lambda_n$, the monic Macdonald-Koornwinder polynomial is given by*

$$P_\lambda(z_1, \dots, z_n; q, t, \mathbf{t}) = \sum_{\substack{\mu^{(i)} \in \Lambda_i, i=1, \dots, n \\ \mu^{(1)} \preceq \mu^{(2)} \preceq \dots \preceq \mu^{(n)} = \lambda}} \prod_{1 \leq i \leq n} P_{\mu^{(i)}/\mu^{(i-1)}}(z_i; q, t, \mathbf{t}), \quad (3.4)$$

where $P_{\mu^{(1)}/\mu^{(0)}}(z; q, t, \mathbf{t}) := P_{\mu^{(1)}}(z; q, t, \mathbf{t})$ (3.2a), (3.2b) by convention.

This is the analog of a classic formula for the usual permutation-symmetric Macdonald polynomials in terms of semistandard tableaux, cf. e.g. [M1, Ch. VI.7] and [LW, Sec. 1].

Remark 2. It follows from the proof in Section 4 below that the branching formula in Theorem 1 holds in fact for any $m \geq \lambda_1$, i.e. the expressions for the branching coefficients $B_{\lambda/\mu}^k$ in Eq. (3.1d) do not depend on $m \geq \lambda_1$.

Remark 3. For $x = t_0 q^h$, $h = 0, 1, 2, \dots$, the degree of the branching polynomials $P_{\lambda/\mu}(x; q, t, \mathbf{t})$ (3.1b) remains bounded by h as the sum in question truncates beyond $k = h$. In particular, for $x = t_0$ only the first (constant) term survives and the complexity of the branching coefficients reduces considerably (cf. [R, p. 100]):

$$P_{\lambda/\mu}(t_0; q, t, \mathbf{t}) = B_{\lambda/\mu}^0(q, t, \mathbf{t}) = (-1)^{|\lambda| - |\mu|} C_{n^m - \mu', m}^{(n+1)^m - \lambda', m}(t, q, \mathbf{t}). \quad (3.5)$$

4. PROOF OF THE BRANCHING FORMULA

Let

$$\prod(x_1, \dots, x_m; z_1, \dots, z_n) := \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \langle x_i; z_j \rangle. \quad (4.1)$$

Mimachi's Cauchy formula [M, Thm. 2.1] states that this kernel expands in terms of Macdonald-Koornwinder polynomials as:

$$\prod(x_1, \dots, x_m; z_1, \dots, z_n) = \sum_{\lambda \subset n^m} (-1)^{mn - |\lambda|} P_\lambda(x_1, \dots, x_m; q, t, \mathbf{t}) P_{m^n - \lambda'}(z_1, \dots, z_n; t, q, \mathbf{t}). \quad (4.2)$$

A similar expansion of the kernel at issue in terms of Okounkov's hyperoctahedral interpolation polynomials is given by the Cauchy formula in [O, Thm. 6.2] (with shifted variables as in [R, Thm. 3.16]). For $n = 1$, the latter Cauchy formula becomes of the form [KNS, Lem. 5.1]:

$$\prod(x_1, \dots, x_m; z) = \sum_{0 \leq r \leq m} (-1)^{m-r} E_r(x_1, \dots, x_m; t, t_0) \langle z; t_0 \rangle_{t, m-r}. \quad (4.3)$$

By expanding the first two factors of the trivial identity

$$\prod (x_1, \dots, x_m; z_1, \dots, z_n, z) = \prod (x_1, \dots, x_m; z_1, \dots, z_n) \prod (x_1, \dots, x_m; z)$$

by means of Mimachi's Cauchy formula (4.2) and the last factor by means of the 'column-row' case of Okounkov's Cauchy formula in Eq. (4.3), one arrives at the equality

$$\begin{aligned} & \sum_{\lambda \subset (n+1)^m} (-1)^{m(n+1)-|\lambda|} P_\lambda(x_1, \dots, x_m; q, t, \mathbf{t}) P_{m^{n+1}-\lambda'}(z_1, \dots, z_n, z; t, q, \mathbf{t}) \\ &= \sum_{\substack{\mu \subset n^m \\ 0 \leq r \leq m}} (-1)^{m(n+1)-|\mu|-r} \left(P_{m^n-\mu'}(z_1, \dots, z_n; t, q, \mathbf{t}) \langle z; t_0 \rangle_{t, m-r} \right. \\ & \quad \left. \times E_r(x_1, \dots, x_m; t, t_0) P_\mu(x_1, \dots, x_m; q, t, \mathbf{t}) \right). \end{aligned}$$

Upon rewriting the RHS with the aid of the Pieri formula (2.3)

$$\begin{aligned} &= \sum_{\substack{\mu \subset n^m \\ 0 \leq r \leq m}} (-1)^{m(n+1)-|\mu|-r} \left(P_{m^n-\mu'}(z_1, \dots, z_n; t, q, \mathbf{t}) \langle z; t_0 \rangle_{t, m-r} \right. \\ & \quad \left. \times \sum_{\substack{\lambda \subset (n+1)^m \\ \lambda \sim_r \mu}} C_{\mu, r}^{\lambda, m}(q, t, \mathbf{t}) P_\lambda(x_1, \dots, x_m; q, t, \mathbf{t}) \right), \end{aligned}$$

and reordering the sums

$$\begin{aligned} &= \sum_{\lambda \subset (n+1)^m} (-1)^{m(n+1)-|\lambda|} \left(P_\lambda(x_1, \dots, x_m; q, t, \mathbf{t}) \right. \\ & \quad \left. \times \sum_{\substack{\mu \subset n^m, 0 \leq r \leq m \\ \mu \sim_r \lambda}} (-1)^{r+|\lambda|-|\mu|} C_{\mu, r}^{\lambda, m}(q, t, \mathbf{t}) P_{m^n-\mu'}(z_1, \dots, z_n; t, q, \mathbf{t}) \langle z; t_0 \rangle_{t, m-r} \right), \end{aligned}$$

one deduces by comparing with the LHS that for any $\lambda \subset (n+1)^m$:

$$\begin{aligned} & P_{m^{n+1}-\lambda'}(z_1, \dots, z_n, z; t, q, \mathbf{t}) = \\ & \sum_{\substack{\mu \subset n^m, 0 \leq r \leq m \\ \mu \sim_r \lambda}} (-1)^{r+|\lambda|-|\mu|} C_{\mu, r}^{\lambda, m}(q, t, \mathbf{t}) P_{m^n-\mu'}(z_1, \dots, z_n; t, q, \mathbf{t}) \langle z; t_0 \rangle_{t, m-r}, \end{aligned}$$

i.e. for any $\lambda \subset m^{n+1}$:

$$\begin{aligned} & P_\lambda(z_1, \dots, z_n, z; q, t, \mathbf{t}) = \\ & \sum_{\substack{\mu \subset m^n, 0 \leq r \leq m \\ n^m-\mu' \sim_r (n+1)^m-\lambda'}} (-1)^{m-r+|\lambda|-|\mu|} C_{\mu, r}^{(n+1)^m-\lambda', m}(t, q, \mathbf{t}) P_\mu(z_1, \dots, z_n; q, t, \mathbf{t}) \langle z; t_0 \rangle_{q, m-r}. \end{aligned} \tag{4.4}$$

To finish the proof we invoke the following lemma.

Lemma 5. *Let $\lambda \subset m^{n+1}$ and $\mu \subset m^n$. Then $n^m - \mu' \sim_r (n+1)^m - \lambda'$ iff $\mu \preceq \lambda$ and $m-d \leq r \leq m$ with $d = \{1 \leq j \leq m \mid \lambda'_j = \mu'_j + 1\}$.*

Proof. The statement of the lemma is immediate upon combining the following two properties: (i) $n^m - \mu' \sim_m (n+1)^m - \lambda'$ iff $\mu \preceq \lambda$ and (ii) $n^m - \mu' \sim_r (n+1)^m - \lambda'$ iff $n^m - \mu' \sim_m (n+1)^m - \lambda'$ and $r \geq m-d$.

Firstly, $n^m - \mu' \sim_m (n+1)^m - \lambda' \Leftrightarrow \exists \nu \subset n^m$ such that $(n^m - \mu')/\nu$ and $((n+1)^m - \lambda')/\nu$ are vertical strips $\Leftrightarrow \exists \kappa \subset m^n$ such that $(m^n - \mu)/\kappa$ and $(m^{n+1} - \lambda)/\kappa$ are horizontal strips $\Leftrightarrow \exists \kappa \subset m^n$ such that $(m^n - \kappa)/\mu$ and $\lambda/(m^n - \kappa)$ are horizontal strips $\Leftrightarrow \mu \preceq \lambda$, which proves part (i).

Secondly—assuming that $n^m - \mu' \sim_m (n+1)^m - \lambda'$ and picking $\nu \subset n^m$ such that $(n^m - \mu')/\nu$ and $((n+1)^m - \lambda')/\nu$ are vertical strips with $|(n^m - \mu')/\nu| + |((n+1)^m - \lambda')/\nu|$ minimal—one has that $|(n^m - \mu')/\nu| + |((n+1)^m - \lambda')/\nu| \leq r \Leftrightarrow |\{1 \leq j \leq m \mid \lambda'_j \neq \mu'_j + 1\}| \leq r \Leftrightarrow m - d \leq r$, which completes the proof of part (ii). \square

The upshot is that Eq. (4.4) can be rewritten as

$$P_\lambda(z_1, \dots, z_n, z; q, t, \mathbf{t}) = \sum_{\substack{\mu \subset m^n, \mu \preceq \lambda \\ m-d \leq r \leq m}} (-1)^{m-r+|\lambda|-|\mu|} C_{n^m-\mu', r}^{(n+1)^m-\lambda', m}(t, q, \mathbf{t}) P_\mu(z_1, \dots, z_n; q, t, \mathbf{t}) \langle z; t_0 \rangle_{q, m-r}$$

and Theorem 1 follows (where our choice of picking m equal to λ_1 corresponds to the minimal value of m such that $\lambda \subset m^{n+1}$, cf. Remark 2 at the end of the previous section).

ACKNOWLEDGMENTS

We are grateful to Alexei Borodin and Ivan Corwin for encouraging us to pursue an explicit branching formula for the Macdonald-Koornwinder polynomials.

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